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On the ϵ-variational principle for set-valued mappings☆

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Abstract

In this paper, the well-known Ekeland variational principle is generalized to the case where set-valued mappings are involved. More specifically, the ϵ-efficient points of a vector optimization problem for set-valued mappings are investigated via a vector Ekeland variational principle for set-valued mappings.
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1. Introduction

The well-known Ekeland variational principle has been used in many areas, such as nonlinear analysis and operations research. More specifically, in some situations, it may not be possible to find an exact solution for an optimization problem, or such an exact solution simply does not exist – for example, if the feasible set is not compact. Thus, it is meaningful to look for an approximate solution instead. The Ekeland variational principle is a very useful tool for approximate optimization problems. It is well known that the Ekeland variational principle is equivalent to the Caristi Fixed Point Theorem, to the Drop Theorem and the Petal Theorem (see [1,2]) and that by virtue of these equivalences, it has found interesting applications in the study of geometry of Banach spaces.

Recently, the Ekeland variational principle has been extended to the case of vector valued functions and set-valued mappings in an ordered Hausdorff topological vector space; see [3–8]. In [5], using Dancs–Hegedus–Medvegyev Theorem (see [1]), we proved a general Ekeland variational principle for a half distance vector valued function (see Theorem 2.1). In this paper, I shall use the general Ekeland variational principle for a half distance vector valued function and properties of the so called ξ-function to prove the vector Ekeland variational principle for a set-valued mapping. At the conclusion of this paper, we give a remark so as to explain that when Theorem 3.1 of [6] does not hold, our result may hold.

The rest of the paper is organized as follows: in Section 2, we introduce some basic notation and preliminary results. In Section 3, we discuss a property of the ξ-function and state a vector Ekeland variational principle for set-valued mappings.

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2. Basic definitions and basic results

Let (X, d) be a complete metric space, and let (E, K) be an ordered topological vector space, in which the ordering is induced by K.

Definition 2.1. Let $A \subset E$ be a nonempty subset. An element $y_0 \in A$ is said to be an efficient point with respect to K if there exists no element $y \in A$ such that

$$y_0 \in y + K \setminus \{0\}.$$

Definition 2.2. An element $y_\epsilon \in A$ is said to be an ϵk^0-efficient point of A with respect to $k^0 \in \operatorname{int} K$ and K if there exists no element y of A such that

$$y_\epsilon \in y + K_{\epsilon k^0},$$

where $K_{\epsilon k^0} = \epsilon k^0 + K \setminus \{0\}$.

We shall denote the set of all efficient points of A by $\mathrm{Eff}(A, K)$ and the set of all ϵk^0 efficient points of A by $\mathrm{Eff}(A, K_{\epsilon k^0})$.

Definition 2.3. We say that a vector-valued function $\Phi : X \times X \to E$ is a half distance if the following properties are satisfied:

1. $\Phi(x, x) = 0, \forall x \in X$;
2. $\Phi(x, y) \leq \Phi(x, z) + \Phi(z, y), \forall x, y, z \in X$.

Definition 2.4. Given $e \in \operatorname{int} K$ and $a \in E$, the Gerstewitz function (see [9,10]) $\xi_{ea} : E \to R$ is defined by

$$\xi_{ea}(y) = \min\{t \in R \mid y \in a + te - K\}.$$

By Theorem 2.1 of [11] and Lemmas 3 and 4 of [5], we have the following results:

Lemma 2.1. *Let $k^0 \in \operatorname{int} K$. Then, the following properties hold:*

(i) $\xi_{k^0}(.)$ *is a continuous and strictly monotone function, namely,* $\xi_{k^0}(.)$ *is continuous and*

$$\xi_{k^0}(y_1) > \xi_{k^0}(y_2) \quad \text{if } y_1 - y_2 \in \operatorname{int} K;$$

(ii) ξ_{k^0} *function is subadditive, namely*

$$\xi_{k^0}(y_1 + y_2) \leq \xi_{k^0}(y_1) + \xi_{k^0}(y_2);$$

(iii) $\xi_{k^0}(lk^0) = l, \forall l \in \mathcal{R}$;

(iv) *If* $y_\epsilon \in \mathrm{Eff}(A, K_{\epsilon k^0})$, *then*

$$\xi_{k^0}(y - y_\epsilon) \geq -\xi_{k^0}(\epsilon k^0) = -\epsilon, \quad \forall y \in A,$$

where $\xi_{k^0}(y) = \xi_{k^0 0}(y)$.

Theorem 2.1. *(see Corollary 1 in [5]) Let (X, d) be a metric space and $\Phi : X \times X \to E$ be a half distance. If for an element $k^0 \in \operatorname{int} K$, the following assumptions are satisfied:*

(i) $\forall x \in X$ *the set* $\{y \in X : \Phi(x, y) + k^0 d(x, y) \in -K\}$ *is closed;*

(ii) *there exist* $v_0 \in X$ *and* $w_0 \in E$ *such that* $\Phi(v_0, x) \geq w_0, \forall x \in X$.

Then there exists an $x^* \in \Gamma(v_0)$, *such that*

$$\Phi(x^*, x) + k^0 d(x^*, x) \notin -K, \forall x \in X \setminus \{x^*\},$$

where $\Gamma(v_0) = \{y \in X \mid \Phi(v_0, y) + k^0 d(v_0, y) \in -K\}$.

3. Main result

In this section, we derive a version of a vector Ekeland variational principle for set-valued mappings which has a close connection to the concept of an ϵk^0 efficient point of a vector optimization problem.

Lemma 3.1. *Let $F : X \to 2^E$ be a set-valued mapping and*

$$\phi(x, z) = \min\{\xi_{k^0}(y_2 - y_1) \mid y_1 \in \mathrm{Eff}(F(x), \mathrm{int}\, K), y_2 \in F(z)\}.$$

Suppose that $x_\epsilon \in X$ and

$$\phi(x_\epsilon, x) + \sqrt{\epsilon} d(x_\epsilon, x) > 0, \quad \forall x \in X \text{ and } x \neq x_\epsilon. \tag{1}$$

Then, for any $y_\epsilon \in \mathrm{Eff}(F(x_\epsilon), K)$, we have

$$y_\epsilon \in \mathrm{Eff}(F_{\epsilon k^0}(X), K),$$

where $F_{\epsilon k^0}(x) = F(x) + \sqrt{\epsilon} d(x_\epsilon, x) k^0$.

Proof. Let $y_\epsilon \in \mathrm{Eff}(F(x_\epsilon), K)$, and let us define a function α from E to $\mathcal{R}$ by

$$\alpha(y) = \xi_{k^0}(y - y_\epsilon - \sqrt{\epsilon} d(x_\epsilon, x) k^0).$$

Then, for any $x \in X$ and $y \in F(x)$, we have

$$\alpha(y + \sqrt{\epsilon} d(x_\epsilon, x) k^0) = \xi_{k^0}(y - y_\epsilon) \geq \phi(x_\epsilon, x).$$

From (1), we have

$$\alpha(y + \sqrt{\epsilon} d(x_\epsilon, x) k^0) > -\sqrt{\epsilon} d(x_\epsilon, x), \quad \forall x \in X \text{ and } x \neq x_\epsilon. \tag{2}$$

It follows from Lemma 2.1(i) and (iii) that, for any $e \in K$, we have

$$\alpha(-e + y_\epsilon) = \xi_{k^0}(-e - \sqrt{\epsilon} d(x_\epsilon, x) k^0) \leq -\sqrt{\epsilon} d(x_\epsilon, x). \tag{3}$$

By (2) and (3), we get

$$F_{\epsilon k^0}(X \setminus \{x_\epsilon\}) \bigcap (-K + y_\epsilon) = \emptyset,$$

i.e.,

$$(F_{\epsilon k^0}(X \setminus \{x_\epsilon\}) - y_\epsilon) \bigcap (-K) = \emptyset. \tag{4}$$

Since $y_\epsilon \in \mathrm{Eff}(F(x_\epsilon), K)$, by (4) we have

$$(F_{\epsilon k^0}(X) - y_\epsilon) \bigcap (-K \setminus \{0\}) = \emptyset,$$

which is equivalent to

$$y_\epsilon \in \mathrm{Eff}(F_{\epsilon k^0}(X), K).$$

The proof is thus complete. □

Theorem 3.1. *Let (X, d) be a complete metric space, and let (E, K) be an ordered Hausdorff topological vector space with a nonempty interior* $\mathrm{int}\, K$. *Let $F : X \to 2^E$ be a set-valued mapping satisfying the following conditions:*

(i) *For a given real number $\epsilon > 0$ and for every $x \in X$, the set*

$$\left\{z \in X \mid (F(z) - \mathrm{Eff}(F(x), \mathrm{int}\, K) + \sqrt{\epsilon} d(x, z) k^0) \bigcap (-K) \neq \emptyset\right\}$$

is closed;

(ii) *For every $x \in X$, $F(x)$ is compact;*

(iii) *There exists a $y \in E$ such that*

$$F(x) - y \subset K, \quad \forall x \in X.$$

Then, for any $x^* \in X$ *satisfying* $\mathrm{Eff}(F(x^*), \mathrm{int}\, K) \subset \mathrm{Eff}(F(X), K_{\epsilon k^0})$*, there exist some points* $x_\epsilon \in X$*,* $y_\epsilon \in F(x_\epsilon)$ *and* $y^* \in \mathrm{Eff}(F(x^*), \mathrm{int}\, K)$ *such that*

(1) $y_\epsilon \le y^*$;
(2) $d(x^*, x_\epsilon) \le \sqrt{\epsilon}$;
(3) $y_\epsilon \in \mathrm{Eff}(F(x_\epsilon), K)$;
(4) $y_\epsilon \in \mathrm{Eff}(F_{\epsilon k^0}(X), K)$*, where* $F_{\epsilon k^0}(x) = F(x) + \sqrt{\epsilon} d(x_\epsilon, x) k^0$.

Proof. Since $F(x), \forall x \in X$, is compact, the set $\mathrm{Eff}(F(x), \mathrm{int}\, K), \forall x \in X$, is also compact. Then, by the continuity of ξ_{k^0}, we may introduce a function $\phi : X \times X \to \mathcal{R}$ defined by

$$\phi(x, z) = \min\{\xi_{k^0}(y_2 - y_1) \mid y_1 \in \mathrm{Eff}(F(x), \mathrm{int}\, K), y_2 \in F(z)\}.$$

Now, we prove that $\phi(x, z)$ satisfies the conditions of Theorem 2.1.

Firstly, we prove that, for any $x^* \in X$, $\phi(x^*, .)$ is bounded below on X. Indeed, by given condition (iii), there exists $\bar{y} \in E$ such that

$$F(X) - \bar{y} \subset K.$$

Then, for any $y_2 \in F(z)$ and $y_1 \in \mathrm{Eff}(F(x), \mathrm{int}\, K)$, we have

$$\xi_{k^0}(y_2 - y_1) \ge \xi_{k^0}(\bar{y} - y_1) \ge \min\{\xi_{k^0}(\bar{y} - y) \mid y \in \mathrm{Eff}(F(x), \mathrm{int}\, K)\}. \tag{5}$$

Suppose that $w_{x^*} = \min\{\xi_{k^0}(\bar{y} - y) \mid y \in \mathrm{Eff}(F(x^*), \mathrm{int}\, K)\}$. It follows from (5) that, for any fixed $x^* \in X$,

$$\phi(x^*, z) \ge w_{x^*}, \quad \forall z \in X.$$

Secondly, we prove that $\phi(x, z)$ is subadditive on $X \times X$. Take any $y_1 \in \mathrm{Eff}(F(x), \mathrm{int}\, K)$, $y_2 \in F(z)$ and $y_3 \in F(v)$. From Lemma 2.1(ii), we have

$$\xi_{k^0}(y_2 - y_1) \le \xi_{k^0}(y_2 - y_3) + \xi_{k^0}(y_3 - y_1). \tag{6}$$

It follows from (6) that

$$\phi(x, z) \le \xi_{k^0}(y_2 - y_3) + \xi_{k^0}(y_3 - y_1).$$

By the arbitrariness of y_1, y_2 and y_3, we get

$$\phi(x, z) \le \min\{\xi_{k^0}(y_2 - y_3) \mid y_2 \in F(x), y_3 \in F(v)\} + \min\{\xi_{k^0}(y_3 - y_1) \mid y_1 \in \mathrm{Eff}(F(x), \mathrm{int}\, K), y_3 \in F(v)\}. \tag{7}$$

It follows that

$$\begin{aligned} \min\{\xi_{k^0}(y_2 - y_3) \mid y_2 \in F(y), y_3 \in F(v)\} &\le \min\{\xi_{k^0}(y_2 - y_3) \mid y_3 \in \mathrm{Eff}(F(v), \mathrm{int}\, K), y_2 \in F(y)\} \\ &= \phi(z, v). \end{aligned} \tag{8}$$

Thus, by (7) and (8), we have

$$\phi(x, z) \le \phi(x, v) + \phi(v, z).$$

Thirdly, we prove that, for every $x \in X$, $\{z \in X \mid \phi(x, z) + \sqrt{\epsilon} d(x, z) \le 0\}$ is closed. Suppose that $A_x = \{z \in X \mid \phi(x, z) + \sqrt{\epsilon} d(x, z) \le 0\}$ and $B_x = \{z \in X \mid (F(z) - \mathrm{Eff}(F(x), \mathrm{int}\, K) + \sqrt{\epsilon} d(x, z) k^0) \bigcap (-K) \ne \emptyset\}$. By the given condition (i), we only need to prove that, for every $x \in X$,

$$A_x = B_x.$$

Suppose that $z \in B_x$. Then, there exist $y_1 \in \mathrm{Eff}(F(x), \mathrm{int}\, K)$, $y_2 \in F(z)$ and $e \in K$, such that

$$y_2 - y_1 + \sqrt{\epsilon} d(x, z) k^0 = -e.$$

It follows from Lemma 2.1(ii) and (iii) that

$$\xi_{k^0}(y_2 - y_1) = \xi_{k^0}(-\sqrt{\epsilon} d(x, z) k^0 - e) \le -\sqrt{\epsilon} d(x, z) + \xi_{k^0}(-e) \le -\sqrt{\epsilon} d(x, z).$$

Then,

$$\phi(x, z) + \sqrt{\epsilon} d(x, z) \leq 0,$$

i.e., $z \in A_x$.

Conversely, suppose that $z \in A_x$. By the compactness of $\mathrm{Eff}(F(x), \mathrm{int}\, K)$ and $F(z)$, there exist $y_1 \in \mathrm{Eff}(F(x), \mathrm{int}\, K)$ and $y_2 \in F(z)$ such that

$$\phi(x, z) = \xi_{k^0}(y_2 - y_1).$$

Then,

$$\xi_{k^0}(y_2 - y_1) + \sqrt{\epsilon} d(x, z) \leq 0. \tag{9}$$

It follows from (9), Lemma 2.1(ii) and Theorem 2.1 in [11] that

$$y_2 - y_1 + \sqrt{\epsilon} d(x, z) k^0 \in -K.$$

Then, we have that $A_x = B_x$, i.e., $\{z \in X \mid \phi(x, z) + \sqrt{\epsilon} d(x, z) \leq 0\}$ is closed.

Thus, we have proved that $\phi(x, z)$ satisfies the conditions of Theorem 2.1.

Now we prove the results of this theorem. By Theorem 2.1, there exists

$$x_\epsilon \in \{z \in X \mid \phi(x^*, z) + \sqrt{\epsilon} d(x^*, z) \leq 0\}$$

such that

$$\phi(x_\epsilon, x) + \sqrt{\epsilon} d(x_\epsilon, x) > 0, \quad x \neq x_\epsilon, \tag{10}$$

and

$$\phi(x^*, x_\epsilon) + \sqrt{\epsilon} d(x^*, x_\epsilon) \leq 0. \tag{11}$$

By the compactness of $\mathrm{Eff}(F(x^*), \mathrm{int}\, K)$ and $F(x_\epsilon)$, there exist $y^* \in \mathrm{Eff}(F(x^*), \mathrm{int}\, K)$ and $y_\epsilon \in F(x_\epsilon)$ such that

$$\phi(x^*, x_\epsilon) = \xi_{k^0}(y_\epsilon - y^*).$$

Moreover, we can take $y_\epsilon \in \mathrm{Eff}(F(x_\epsilon), K)$. Indeed, if $y_\epsilon \notin \mathrm{Eff}(F(x_\epsilon), K)$, by Lemma 2.1 of [12], there exists $y' \in \mathrm{Eff}(F(x_\epsilon), K)$ such that $y' \leq y_\epsilon$. It follows from Lemma 2.1(i) that

$$\xi_{k^0}(y_\epsilon - y^*) \geq \xi_{k^0}(y' - y^*).$$

However, by the definition of $\phi(x^*, x_\epsilon)$, we have

$$\xi_{k^0}(y_\epsilon - y^*) \leq \xi_{k^0}(y' - y^*),$$

i.e.,

$$\xi_{k^0}(y_\epsilon - y^*) = \xi_{k^0}(y' - y^*).$$

Take $y_\epsilon = y'$. Naturally, we have

$$y_\epsilon \in \mathrm{Eff}(F(x_\epsilon), K), \quad \text{and} \quad \phi(x^*, x_\epsilon) = \xi_{k^0}(y_\epsilon - y^*). \tag{12}$$

Since $y^* \in \mathrm{Eff}(F(x^*), \mathrm{int}\, K) \subset \mathrm{Eff}(F(X), K_{\epsilon k^0})$, by Lemma 2.1(iv) we have

$$\xi_{k^0}(y_\epsilon - y^*) \geq -\epsilon. \tag{13}$$

It follows from (11), (13) and Lemma 2.1(ii) that

$$d(x^*, x_\epsilon) \leq \sqrt{\epsilon},$$

and

$$\xi_{k^0}(y_\epsilon - y^* + \sqrt{\epsilon} d(x^*, x_\epsilon) k^0) \leq 0.$$

Then, by Lemma 2.1(i), we have

$$y_\epsilon - y^* + \sqrt{\epsilon} d(x^*, x_\epsilon) k^0 \in -K.$$

Thus,

$$y_\epsilon \leq y^*.$$

From (10), (12) and Lemma 3.1, we have

$$y_\epsilon \in \mathrm{Eff}(F_{\epsilon k^0}(X), K).$$

Thus, this completes the proof. □

Remark 3.1. When $F(x)$ is upper semicontinuous (see [13]) on X and, for each $x \in X$, $F(x)$ is compact, we have that, for a given real number $\epsilon > 0$ and for every $x \in X$, the set

$$\left\{z \in X \mid (F(z) - \mathrm{Eff}(F(x), \mathrm{int}\, K) + \sqrt{\epsilon} d(x,z)k^0) \bigcap (-K) \neq \emptyset\right\}$$

is closed. However, the converse may not hold. For example, suppose that $F : \mathcal{R} \to 2^{\mathcal{R}}$ is defined by

$$F(x) = \begin{cases} 0 & \text{if } x = 0, \\ [0, 1] & \text{if } x \neq 0. \end{cases}$$

Take $K = \mathcal{R}^+$, $\sqrt{\epsilon} = 0.5$ and $k^0 = 0.5$. Then, for every $x \in \mathcal{R}$, the set

$$\left\{z \in X \mid (F(z) - \mathrm{Eff}(F(x), \mathrm{int}\, K) + \sqrt{\epsilon} d(x,z)k^0) \bigcap (-K) \neq \emptyset\right\} = \{x\}$$

is closed. Obviously, $F(.)$ is not upper semicontinuous at $x = 0$. Thus, when Theorem 3.1 in [6] does not hold, our Theorem 3.1 may hold.

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